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LETTER TO THE EDITOR

Solutions of n -simplex equation from solutions of braid group representation*

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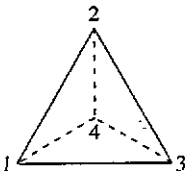
Abstract. It is shown that a kind of solution of the n -simplex equation can be obtained from representations of the braid group. The symmetries in its solution space are also discussed.

Recently much interest has been paid to the investigation of the higher-dimensional integrable systems in the quantum field theory [1] and in the statistical mechanics [2]. For the lower dimension case, the Yang-Baxter equation (YBE) plays a crucial role of which the structure is now fairly well understood. As a substitution of YBE the tetrahedron equation becomes an integrability condition of the exactly-solved model in three dimensions [3], from which the commutativity of the layer-to-layer transfer matrices is preserved. One of the approaches is the n -simplex equation [4] and it is said that the case of $n = 3$ corresponds to the tetrahedron equation. The aim of this letter is to expose some procedure for deriving solutions of the n -simplex equation from braid group representations (i.e. solutions of parameter-independent YBE) [5]. Meanwhile we would like to derive some symmetry transformations in solution space of the 3-simplex equation as an example.

The 3-simplex equation we will consider takes the following form:

$$R_{123}R_{214}R_{341}R_{432} = R_{234}R_{143}R_{412}R_{321} \tag{1}$$

where the order of subscripts are chosen in such a way that the normal of each surface of the 3-simplex is always towards the inside of the 3-simplex (tetrahedron).



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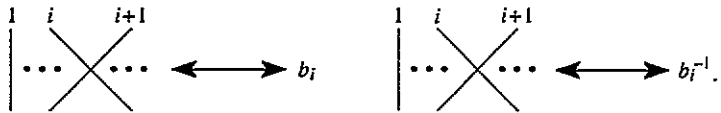
Certainly, the positive direction of the normal is determined by right-hand helicity, for example, (123), (341), The matrices in (1) stands for the scattering of three strings, for example,

$$R_{214}|\mu_1, \mu_2, \mu_3, \mu_4\rangle = \sum_{\nu_1 \nu_2 \nu_3} R_{\mu_2 \mu_1 \mu_4}^{\nu_2 \nu_1 \nu_3} |\nu_1, \nu_2, \mu_3, \nu_4\rangle. \tag{2}$$

Solving solutions of (1) is a complicated problem. It is known that many representations of the braid group have been found in recent years. We will show that if one has a representation of the braid group, one can obtain a kind of solution of the 3-simplex equation. A braid group is a category of the free group under the constraint of the following equivalence relations:

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \\ b_i b_j &= b_j b_i \quad \text{for } |i - j| > 1. \end{aligned} \tag{3}$$

It is called a braid group due to its simple realization on N -strings by the identity



Then the equivalence relation (3) becomes an evident topological equivalence relation. If a representation of braid group is

$$\rho : b_i \rightarrow S_{i,i+1} = I^{(1)} \otimes \dots \otimes I^{(i-1)} \otimes S \otimes I^{(i+2)} \otimes \dots \otimes I^{(N)} \tag{4}$$

where $S \in \text{End}(V \otimes V)$ satisfying the following parameter independent Yang-Baxter equation:

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23}. \tag{5}$$

If we define an operator

$$t := \prod_{i=1}^n \prod_{j=1}^{i-1} b_i$$

which is understood as an ordered product from right to left or vice versa. We can show that the following identity holds:

$$t_1 t_2 t_1 t_2 \dots = t_2 t_1 t_2 t_1 \dots \tag{6}$$

where the number of t 's in the alternative product is $n + 1$. The case $n = 2$ is exactly the elementary equivalence relations of braid group (3). For $n = 3$ we have

$$t_1 t_2 t_1 t_2 = t_2 t_1 t_2 t_1 \tag{7}$$

where $t_1 = b_1 b_2 b_1$ and $t_2 = b_2 b_3 b_2$. Thus, if we know a representation of a braid group, we will have a solution of the following equation:

$$\check{R}_{123} \check{R}_{234} \check{R}_{123} \check{R}_{234} = \check{R}_{234} \check{R}_{123} \check{R}_{234} \check{R}_{123} \tag{8}$$

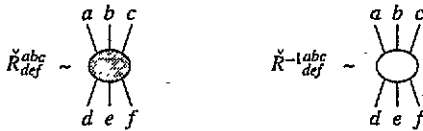
where $\check{R}_{123} := \check{R} \otimes I$, $\check{R}_{234} := I \otimes \check{R}$ and $\check{R} \in \text{End}(V \otimes V \otimes V)$. This is easily realized by

$$\rho : t_1 \rightarrow \check{R}_{123}$$

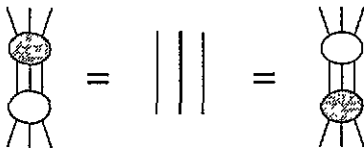
due to $t_1 = b_1 b_2 b_1$ etc, then the following identities holds:

$$\check{R}_{123} = S_{12} S_{23} S_{12} \otimes I \quad \text{etc.} \tag{9}$$

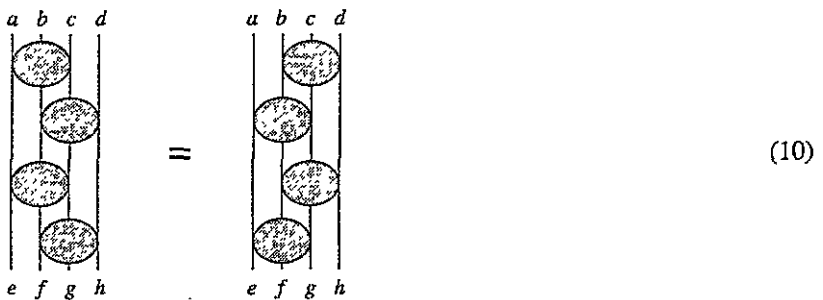
As to (8), one may find some symmetry transformation of it. If one writes out (8) into a component form instead of a matrix form, one can easily find that the equation can be symbolized by a Kauffman diagram. That says if we denote



The inverse relation and (8) are depicted, respectively, as

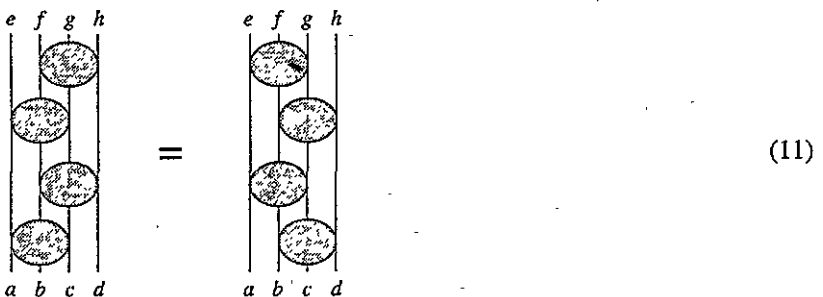


and



where the inner line connecting legs of two shadows implies the summation over the repeated labels on the legs, and a simple vertical line stands for a unit matrix. It is not difficult to find that the diagram (10) has the following symmetries.

Flipping via a horizontal axis, denoted by H




or flipping via a vertical axis denoted by V

or via both in term $VH = HV$.

Thus we have

$$\begin{array}{ccc}
 (7) & \xrightarrow{H} & (8) \\
 V \downarrow & & \downarrow V \\
 (9) & \xrightarrow{H} & (10)
 \end{array}$$

and $H^2 = id, V^2 = id, HV = -VH$. All the four diagrams (10)–(13) depict the same equation (8). So the solution space of (8) has a discrete group symmetry. $\{id, H, V, VH | H^2 = id, V^2 = id, HV = VH\}$. The action of this group brings one solution of (8) into the other three new solutions, i.e. if \check{R}_{def}^{abc} is a solution of (8), then $\check{R}'_{def}{}^{abc} = \check{R}_{fed}^{cha}, \check{R}''_{def}{}^{abc} = \check{R}_{abc}^{edf}$ and $\check{R}'''_{def}{}^{abc} = \check{R}_{cba}^{fed}$ will be solutions of (8).

Furthermore, if giving a direction to the Kauffman diagram $\check{R}_{def}^{abc} \sim$ , and adding

a minus sign to the labels on the tip of the arrow, we can find that the summation of such labels on both sides of the diagram (10) are equal. This brings about a continuous transformation from a solution of (8) into another

$$\check{R}_{def}^{abc} \rightarrow \check{R}'_{def}{}^{abc} = t^{a+b+c-d-e-f} \check{R}_{def}^{abc}. \tag{14}$$

Starting from the matrix form of (8), we can obtain two more continuous transformations in solution space. They are an overall factor transformation $\check{R} \rightarrow \tau \check{R}$; a similar transformation by a tensor product of matrices $\check{R} \rightarrow (\Lambda \otimes \Lambda \otimes \Lambda) \check{R} (\Lambda^{-1} \otimes \Lambda^{-1} \otimes \Lambda^{-1})$. Because eigenvalues of a matrix are invariant under a similar transformation, the latter is a transformation within the subset of solution space, which is specified by the eigenvalues of \check{R} .

In the above we have discussed (8) in detail. Now we introduce a new R -matrix

$$\check{R} = RP \quad (15)$$

where P is defined as

$$P|\mu_1, \mu_2, \mu_3\rangle := |\mu_3, \mu_2, \mu_1\rangle.$$

Then we can show the R -matrix satisfying the following equation as long as the \check{R} -matrix satisfies (8):

$$R_{123}R_{214}R_{341}R_{432} = R_{234}R_{143}R_{412}R_{321} \quad (16)$$

which is a variant of the FM 3-simplex equation we have introduced at the beginning of our discussion.

In a similar way, one may discuss the case of a 4-simplex equation and so on. The key point is that (6) is an identity on the braid group, then if one has a representation of the braid group, one can write down an expression from the expression of t_i on the basis of S -matrix, which is supposed to be a solution of the parameter-independent Yang–Baxter equation.

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